

A large deviation principle for weighted Riesz interactions

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Abstract

We prove a large deviation principle for the sequence of push-forwards of empirical measures in the setting of Riesz potential interactions on compact subsets K in \mathbb{R}^d with continuous external fields. Our results are valid for base measures on K satisfying a strong Bernstein-Markov type property for Riesz potentials. Furthermore, we give sufficient conditions on K (which are satisfied if K is a smooth submanifold) so that a measure on K which satisfies a mass-density condition will also satisfy this strong Bernstein-Markov property.

1 Introduction

Fix a positive integer $d > 2$. Let $W(y) = \frac{1}{|y|^\alpha}$ where $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, $|y|^2 = \sum_{j=1}^d y_j^2$, and $0 < \alpha < d$. For $K \subset \mathbb{R}^d$ a compact set of positive Riesz α -capacity and $Q : K \rightarrow \mathbb{R}$ continuous, we consider the ensemble of probability measures $Prob_n$ on K^n :

$$Prob_n := \frac{1}{Z_n} \exp \left[- \sum_{1 \leq i \neq j \leq n} W(x_i - x_j) \right] \cdot \exp \left[-2n \sum_{j=1}^n Q(x_j) \right] d\nu(x_1) \cdots d\nu(x_n) \quad (1.1)$$

where $d\nu$ is a measure on K and Z_n is a normalizing constant. Our main result, stated at the end of the introduction, is a large deviation principle for the sequence $\{\sigma_n = (j_n)_*(Prob_n)\}$ of probability measures on $\mathcal{M}(K)$, the space of probability measures on K , where $j_n : K^n \rightarrow \mathcal{M}(K)$ is the empirical measure map $j_n(x_1, \dots, x_n) = \frac{1}{n} \sum_{j=1}^n \delta_{x_j}$. Weighted Riesz interactions have been studied by many authors, e.g., [5] and [10], but generally the situation considered is $K = \mathbb{R}^d$ and $d\nu$ is Lebesgue measure (and $Q(x)$ satisfies a growth condition as $|x| \rightarrow \infty$).

In this paper, we follow the technique utilized, e.g., in [2] and [3]. We first discuss weighted Riesz potential-theoretic notions in the next section such as the weighted Riesz energy functional I^Q . This entails a weighted Riesz energy minimization problem $\inf_{\mu \in \mathcal{M}(K)} I^Q(\mu)$ with minimizer $\mu_{K,Q}$ and a corresponding discretization. Section 3 forms the heart of the paper; there we prove a Bernstein-type estimate (Proposition 3.3) on

“polynomial-like” functions arising from our discretization process. This leads to a sufficient mass-density condition on a measure μ on certain compact sets $K \subset \mathbb{R}^d$ so that we have a good comparability between supremum norms and $L^1(\mu)$ norms of weighted versions of such functions (Theorem 3.4 on strong Bernstein-Markov measures for Riesz potentials on K ; see Remark 3.5 for this definition). With these ingredients in hand, the consequences, such as one-point correlation asymptotics (Corollary 4.4) and a large deviation principle, follow:

Theorem 1.1. *Let ν be a strong Bernstein-Markov measure for K and Q continuous on K . The sequence $\{\sigma_n = (j_n)_*(\text{Prob}_n)\}$ of probability measures on $\mathcal{M}(K)$ satisfies a large deviation principle with speed n^2 and good rate function $\mathcal{I} := \mathcal{I}_{K,Q}$ where, for $\mu \in \mathcal{M}(K)$,*

$$\mathcal{I}(\mu) = I^Q(\mu) - I^Q(\mu_{K,Q}).$$

2 Riesz potential theory

Let $K \subset \mathbb{R}^d$ be compact and let $\mathcal{M}(K)$ be the set of probability measures on K endowed with the topology of weak convergence. Fix $0 < \alpha < d$. We consider the Riesz energy minimization problem:

$$\inf_{\mu \in \mathcal{M}(K)} I(\mu)$$

where

$$I(\mu) := \int_K \int_K \frac{1}{|x - y|^\alpha} d\mu(x) d\mu(y)$$

is the Riesz energy of μ . We will restrict to $0 < \alpha < d$ for the rest of the paper. If there exists $\mu \in \mathcal{M}(K)$ with $I(\mu) < \infty$ we say K has *positive Riesz α -capacity* (and henceforth we drop the “ α ”). We remark that if one considers the cone \mathcal{M}_+ of all positive measures on \mathbb{R}^d (not necessarily with compact support), it is known (cf., [9, Chapter I], [5]) that

1. for $\mu \in \mathcal{M}_+$, $I(\mu) \geq 0$;
2. $I(\mu) = 0$ if and only if $\mu = 0$;
3. $\mu \rightarrow I(\mu)$ is strictly convex on $\{\mu \in \mathcal{M}_+ : I(\mu) < \infty\}$.

Note that Landkof [9] replaces our α with $d - \alpha$.

Remark 2.1. We add for future use that 2. holds for signed measures $\mu = \mu_1 - \mu_2$ with $\mu(K) = 0$ when the mixed energy $I(\mu_1, \mu_2) := \int_K \int_K \frac{1}{|x - y|^\alpha} d\mu_1(x) d\mu_2(y)$ is finite; this is in Theorem 1.15, p. 79 of [9].

We also consider a weighted Riesz energy minimization problem. Given a compact set K of positive Riesz capacity, and a lower semicontinuous function Q on K with $\{x \in K : Q(x) < \infty\}$ of positive Riesz capacity (we write $Q \in \mathcal{A}(K)$), we consider

$$\inf_{\mu \in \mathcal{M}(K)} I^Q(\mu)$$

where

$$I^Q(\mu) := I(\mu) + 2 \int_K Q(x) d\mu(x) := \int_K \int_K \frac{1}{|x - y|^\alpha} d\mu(x) d\mu(y) + 2 \int_K Q(x) d\mu(x).$$

In later portions of this paper, we will restrict to $Q \in C(K)$ (continuous functions on K).

Remark 2.2. In [5] the authors consider the situation where $K = \mathbb{R}^d$ and $Q(x)$ satisfies a growth condition as $|x| \rightarrow \infty$. Their Theorem 1.2 gives general results, in this setting, for the weighted energy minimization problems, while their Theorem 1.1 is a large deviation principle using $Prob_n$ measures as in (1.1) which are taken with respect to Lebesgue measure on \mathbb{R}^d . We will allow general (possibly singular) measures ν in (1.1) for our large deviation principle, Theorem 5.7. See also [10] for further results.

We define the Riesz potential associated to a positive measure μ on K :

$$U^\mu(x) := \int_K \frac{1}{|x - y|^\alpha} d\mu(y).$$

The following properties hold:

1. The Riesz potential is a lower semicontinuous function on \mathbb{R}^d . It is superharmonic in \mathbb{R}^d for $0 < \alpha \leq d - 2$ and subharmonic outside K for $d - 2 < \alpha$, ([9, Theorem I.1.4 p. 66]).
2. For $d - 2 < \alpha < d$, we have a *domination principle* ([9], Theorem 1.2.9, p. 115): for μ a measure whose potential U^μ is finite μ -a.e., and u a superharmonic function, if the inequality $U^\mu \leq u$ holds μ -a.e., then it holds everywhere.
3. Also, for $d - 2 \leq \alpha < d$, we have a *maximum principle* ([9], Theorem 1.10, p. 71): for μ a measure with $U^\mu \leq M$ μ -a.e., this estimate holds everywhere.
4. There is a *weak maximum principle* ([9], Theorem 1.5, p. 66): for all $0 < \alpha < d$, given μ a measure with $U^\mu \leq M$ on $\text{supp}(\mu)$, we have $U^\mu \leq 2^\alpha M$ on \mathbb{R}^d .
5. This last property is sufficient to prove a *continuity property* of Riesz potentials ([9], Theorem 1.7, p. 69): for all $0 < \alpha < d$, given μ a measure with U^μ continuous on $\text{supp}(\mu)$, we have U^μ is continuous on \mathbb{R}^d .

Throughout, unless otherwise specified, we assume $0 < \alpha < d$. Following the arguments on pp. 27-33 in [11] for weighted logarithmic potential theory (or for $K = \mathbb{R}^d$ in [5]) we have the following.

Theorem 2.3. For $K \subset \mathbb{R}^d$ compact and of positive Riesz capacity, and for $Q \in \mathcal{A}(K)$,

1. $V_w := \inf_{\mu \in \mathcal{M}(K)} I^Q(\mu)$ is finite;
2. there exists a unique weighted equilibrium measure $\mu_{K,Q} \in \mathcal{M}(K)$ with $I^Q(\mu_{K,Q}) = V_w$;

3. the support $S_w := \text{supp}(\mu_{K,Q})$ is contained in $\{x \in K : Q(x) < \infty\}$ and S_w is of positive Riesz capacity;
4. if we let $F_w := V_w - \int_K Q(x) d\mu_{K,Q}(x)$, then

$$U^{\mu_{K,Q}}(x) + Q(x) \geq F_w \text{ on } K \setminus P \text{ where } P \text{ is of zero Riesz capacity (possibly empty);}$$

$$U^{\mu_{K,Q}}(x) + Q(x) \leq F_w \text{ for all } x \in S_w.$$

Remark 2.4. In the proof of the Frostman-type property 4. in [11], one simply replaces “q.e.” – off of a set of positive logarithmic capacity in \mathbb{C} – by “off of a set of zero Riesz capacity” as the essential property used is the existence of a measure of finite logarithmic energy on a compact subset of a set of positive logarithmic capacity in \mathbb{C} .

Remark 2.5. For $K \subset \mathbb{R}^d$ compact and of positive Riesz capacity, if $\mu \in \mathcal{M}(K)$ with $I(\mu) < \infty$, one can consider a weighted energy minimization problem with the *upper semicontinuous* weight $Q = -U^\mu$. Following the proof of Lemma 5.1 of [3], the minimum is attained (uniquely) by the measure μ ; i.e.,

$$I(\mu) + 2 \int_K Q(x) d\mu(x) \leq I(\nu) + 2 \int_K Q(x) d\nu(x) \text{ for all } \nu \in \mathcal{M}(K)$$

with equality if and only if $\nu = \mu$. This uses Remark 2.1.

The “converse” to 4. of Theorem 2.3 holds as well. This is stated/proved in [5] in their setting (Theorem 1.2 (1.10) and (1.11)).

Proposition 2.6. *Let $K \subset \mathbb{R}^d$ be compact and of positive Riesz capacity and let $Q \in \mathcal{A}(K)$. For a measure $\mu \in \mathcal{M}(K)$, if there exists a constant C such that*

$$U^\mu(x) + Q(x) \geq C \text{ on } K \setminus P \text{ where } P \text{ is of zero Riesz capacity}$$

and

$$U^\mu(x) + Q(x) \leq C \text{ for all } x \in \text{supp}\mu,$$

then $\mu = \mu_{K,Q}$.

Proof. We write

$$\mu_{K,Q} = \mu + (\mu_{K,Q} - \mu).$$

Then

$$I^Q(\mu) \geq I^Q(\mu_{K,Q}) = I^Q(\mu) + I(\mu_{K,Q} - \mu) + 2R$$

with

$$\begin{aligned} R &:= \int_K \left[\int_K \frac{1}{|x-y|^\alpha} d\mu(y) + Q(x) \right] d(\mu_{K,Q} - \mu)(x) \\ &= \int_K (U^\mu(x) + Q(x)) d(\mu_{K,Q} - \mu)(x). \end{aligned}$$

Note that the above computation is justified. Indeed, from the assumptions $I^Q(\mu) < \infty$ and μ has compact support, the quantities $I^Q(\mu)$, $I(\mu)$, $\int Q d\mu$, and the mixed energy $I(\mu, \mu_{K,Q})$ are all finite. Making use of the inequalities in the hypotheses, we conclude that

$$R \geq C \int_K d\mu_{K,Q} - C \int_K d\mu = 0.$$

Recall that $I(\mu_{K,Q} - \mu) \geq 0$ with equality if and only if $\mu_{K,Q} = \mu$ (Remark 2.1). Thus

$$I^Q(\mu) \geq I^Q(\mu_{K,Q}) \geq I^Q(\mu)$$

so that equality holds throughout, and $I^Q(\mu) = I^Q(\mu_{K,Q})$, from which follows $\mu = \mu_{K,Q}$. \square

Remark 2.7. For $d-2 \leq \alpha < d$, since we have a maximum principle, using 4. of Theorem 2.3 and following the argument in the proof of Theorem 4 in [4], for any $\mu \in \mathcal{M}(K)$ we have

$$“\inf_{x \in S_w}” [U^\mu(x) + Q(x)] \leq F_w$$

and

$$\sup_{x \in \text{supp} \mu} [U^\mu(x) + Q(x)] \geq F_w.$$

Here, “ $\inf_{x \in S}” F(x)$ ” denotes the largest number L such that on S the real-valued function F takes values smaller than L only on a set of zero Riesz capacity. The corresponding version of this result for certain weights on all of \mathbb{R}^d is stated as equations (1.14) and (1.15) in [5].

As in [11], we can characterize the compact sets $K \subset \mathbb{R}^d$ which arise as supports of a weighted energy minimizing measure.

Theorem 2.8. *Let $K \subset \mathbb{R}^d$ be compact and of positive Riesz capacity at each point of K . Then there exists $Q \in \mathcal{A}(K)$ such that $S_w = K$.*

Proof. Suppose we can find a probability measure μ with support K such that U^μ is continuous on K (and hence continuous on \mathbb{R}^d by the aforementioned continuity property of Riesz potentials in [9], Theorem 1.7, p. 69). Then taking $Q(x) := -U^\mu(x)$ on K , we have

$$U^\mu(x) + Q(x) = 0 \text{ on } K$$

so that by Proposition 2.6 we have $\mu = \mu_{K,Q}$.

To construct such a μ , we follow the arguments in Lemma I.6.10 and Corollary I.6.11 in [11] (we do not need the final statement in Lemma I.6.10). In particular, for any compact $S \subset \mathbb{R}^d$ of positive Riesz capacity, we obtain a finite, positive measure ν with support in S such that U^ν is continuous. Using this, we follow exactly the proof of Theorem IV.1.1 in [11]. \square

Next we discretize: for $n \geq 2$, let

$$\begin{aligned} VDM_n^Q(x_1, \dots, x_n) &:= \exp \left[- \sum_{1 \leq i \neq j \leq n} W(x_i - x_j) \right] \cdot \exp \left[-2n \sum_{j=1}^n Q(x_j) \right] \\ &= \exp \left[- \sum_{1 \leq i \neq j \leq n} \frac{1}{|x_i - x_j|^\alpha} \right] \cdot \exp \left[-2n \sum_{j=1}^n Q(x_j) \right] =: \exp[-L_n(x_1, \dots, x_n)] \\ \text{where } L_n(x_1, \dots, x_n) &= \sum_{1 \leq i \neq j \leq n} \frac{1}{|x_i - x_j|^\alpha} + 2n \sum_{j=1}^n Q(x_j). \end{aligned}$$

Thus

$$\frac{1}{n(n-1)} L_n(x_1, \dots, x_n) = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} \frac{1}{|x_i - x_j|^\alpha} + \frac{2}{n-1} \sum_{j=1}^n Q(x_j)$$

is the approximate weighted Riesz energy of $\mu_n := \frac{1}{n} \sum_{j=1}^n \delta_{x_j}$; i.e., where we ignore “diagonal” terms (which make the true Riesz energy of μ_n infinite).

We define the n -th weighted diameter $\delta_n^Q(K)$ by

$$\delta_n^Q(K) := \sup_{x_1, \dots, x_n \in K} VDM_n^Q(x_1, \dots, x_n)^{1/n^2}. \quad (2.1)$$

We will show the limit of these quantities exists, and this *weighted transfinite diameter of K with respect to Q* satisfies

$$\delta^Q(K) := \lim_{n \rightarrow \infty} \delta_n^Q(K) = \exp(-V_w) = \exp(-I^Q(\mu_{K,Q})).$$

By upper semicontinuity of $(x_1, \dots, x_n) \rightarrow -L_n(x_1, \dots, x_n)$ on K^n and $-Q$ on K the supremum in (2.1) is attained; we call any collection of n points of K at which the maximum is attained *weighted Fekete points* of order n for K, Q . Following the proofs of Propositions 3.1–3.3 of [2, Section 3] we have:

Theorem 2.9. *Given $K \subset \mathbb{R}^d$ compact and of positive Riesz capacity and $Q \in \mathcal{A}(K)$,*

1. *if $\{\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{x_j^{(n)}}\} \subset \mathcal{M}(K)$ converge weakly to $\mu \in \mathcal{M}(K)$, then*

$$\limsup_{n \rightarrow \infty} VDM_n^Q(x_1^{(n)}, \dots, x_n^{(n)})^{1/n^2} \leq \exp(-I^Q(\mu)); \quad (2.2)$$

2. *we have $\delta^Q(K) := \lim_{n \rightarrow \infty} \delta_n^Q(K)$ exists and*

$$\delta^Q(K) = \exp(-V_w) = \exp(-I^Q(\mu_{K,Q}));$$

3. if $\{x_j^{(n)}\}_{j=1,\dots,n; n=2,3,\dots} \subset K$ and

$$\lim_{n \rightarrow \infty} VDM_n^Q(x_1^{(n)}, \dots, x_n^{(n)})^{1/n^2} = \exp(-V_w)$$

then

$$\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{x_j^{(n)}} \rightarrow \mu_{K,Q} \text{ weakly.}$$

Proof. To prove 1., note that $\mu_n \times \mu_n \rightarrow \mu \times \mu$ in $\mathcal{M}(K) \times \mathcal{M}(K)$. Exploiting lower semicontinuity of the kernel

$$(x, y) \rightarrow W(x - y) = \frac{1}{|x - y|^\alpha},$$

for $M \in \mathbb{R}$ we define the continuous kernel

$$h_M(x, y) := \min[M, W(x - y)]$$

and we have

$$\begin{aligned} I^Q(\mu) &= \lim_{M \rightarrow \infty} \int_K \int_K h_M(x, y) d\mu(x) d\mu(y) + 2 \int_K Q d\mu \\ &\leq \lim_{M \rightarrow \infty} \liminf_{n \rightarrow \infty} \left(\int_K \int_K h_M(x, y) d\mu_n(x) d\mu_n(y) + 2 \int_K Q d\mu_n \right) \\ &\leq \lim_{M \rightarrow \infty} \liminf_{n \rightarrow \infty} \left[\frac{1}{n^2} (nM + \sum_{j \neq k} \frac{1}{|x_j^{(n)} - x_k^{(n)}|^\alpha}) + \frac{2}{n} \sum_{j=1}^n Q(x_j^{(n)}) \right]. \end{aligned}$$

Here we used lower semicontinuity of Q to conclude $\int_K Q d\mu \leq \liminf_{n \rightarrow \infty} \int_K Q d\mu_n$. Thus

$$\limsup_{n \rightarrow \infty} VDM_n^Q(x_1^{(n)}, \dots, x_n^{(n)})^{1/n^2} \leq e^{-I^Q(\mu)}.$$

To prove 2., let

$$D_n(K) := \inf_{x_1, \dots, x_n \in K} \frac{1}{n(n-1)} L_n(x_1, \dots, x_n).$$

Note that

$$D_n(K) \leq \frac{-1}{n(n-1)} \log VDM_n^Q(x_1, \dots, x_n), \text{ all } (x_1, \dots, x_n) \in K^n \quad (2.3)$$

with equality for weighted Fekete points of order n for K, Q . Fix n points $x_1, \dots, x_n \in K$. Then

$$D_n(K) \leq \frac{1}{n(n-1)} L_n(x_1, \dots, x_n) = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} \frac{1}{|x_i - x_j|^\alpha} + \frac{2}{n-1} \sum_{j=1}^n Q(x_j).$$

For $\mu \in \mathcal{M}(K)$, integrate both sides with respect to $\prod_{i < j} d\mu(x_i) d\mu(x_j)$:

$$D_n(K) \leq I(\mu) + \frac{2n}{n-1} \int_K Q d\mu.$$

Thus

$$\limsup_{n \rightarrow \infty} D_n(K) \leq I^Q(\mu). \quad (2.4)$$

On the other hand, taking $\mu_n := \frac{1}{n} \sum_{j=1}^n \delta_{x_j^{(n)}}$ where $D_n(K) = \frac{1}{n(n-1)} L_n(x_1^{(n)}, \dots, x_n^{(n)})$ (i.e., weighted Fekete points of order n for K, Q), if μ is any weak limit of this sequence then 1. implies that

$$\limsup_{n \rightarrow \infty} VDM_n^Q(x_1^{(n)}, \dots, x_n^{(n)})^{1/n^2} \leq \exp(-I^Q(\mu)).$$

From (2.4) and the equality portion of (2.3),

$$\limsup_{n \rightarrow \infty} \frac{-1}{n(n-1)} \log VDM_n^Q(x_1^{(n)}, \dots, x_n^{(n)}) = \limsup_{n \rightarrow \infty} [-D_n(K)] \leq I^Q(\mu). \quad (2.5)$$

Thus $\lim_{n \rightarrow \infty} VDM_n^Q(x_1^{(n)}, \dots, x_n^{(n)})^{1/n^2}$ exists and equals $\exp(-I^Q(\mu))$ where μ is any weak limit of weighted Fekete measures. Since $\mu_{K,Q}$ is the unique weighted energy minimizing measure we claim that 2. follows: for if $\sigma \in \mathcal{M}(K)$ is arbitrary, applying 1. to σ and using (2.5) shows

$$\begin{aligned} \exp(-I^Q(\sigma)) &\leq \liminf_{n \rightarrow \infty} VDM_n^Q(x_1^{(n)}, \dots, x_n^{(n)})^{1/n^2} \\ &\leq \limsup_{n \rightarrow \infty} VDM_n^Q(x_1^{(n)}, \dots, x_n^{(n)})^{1/n^2} \leq \exp(-I^Q(\mu)). \end{aligned}$$

Item 3. follows from 1. and 2. □

Remark 2.10. Note that item 1. required lower semicontinuity of the kernel

$$(x, y) \rightarrow W(x - y) = \frac{1}{|x - y|^\alpha}$$

and of Q while item 2. (and hence 3.) required the existence and uniqueness of the weighted energy minimizing measure $\mu_{K,Q}$.

3 Bernstein-type estimate

In this section, we always assume $Q \in C(K)$.

If we fix $n - 1$ points $x_2, \dots, x_n \in K$ and consider

$$y \rightarrow VDM_n^Q(y, x_2, \dots, x_n), \quad y = (y_1, \dots, y_d) \in \mathbb{R}^d,$$

then this function is of the form

$$g(x_2, \dots, x_n) \cdot f_n^Q(y) := g(x_2, \dots, x_n) \cdot \exp \left[- \sum_{j=2}^n \frac{1}{|y - x_j|^\alpha} - 2nQ(y) \right]$$

where

$$g(x_2, \dots, x_n) = \exp \left(-2 \sum_{j=2}^n Q(x_j) \right) \cdot \exp \left(- \sum_{2 \leq j \neq k \leq n} \frac{1}{|x_k - x_j|^\alpha} \right).$$

For notation, fixing K and Q , we let

$$\mathcal{P}_n^Q := \left\{ f_n^Q(y) := \exp \left(- \sum_{j=2}^n \frac{1}{|y - x_j|^\alpha} - 2nQ(y) \right) : x_2, \dots, x_n \in K \right\}. \quad (3.1)$$

If $Q \equiv 0$, we simply write \mathcal{P}_n and f_n .

We recall the definition of the box-counting (or Minkowski) dimension of a bounded subset K of \mathbb{R}^d , see e.g. [7, Chapter 3]. Let $N_\delta(K)$ be the smallest number of closed balls of radius δ which can cover K . The lower and upper box-counting dimensions of K are defined as

$$\underline{\dim}_B K = \liminf_{\delta \rightarrow 0} \frac{\log N_\delta(K)}{-\log \delta}, \quad \overline{\dim}_B K = \limsup_{\delta \rightarrow 0} \frac{\log N_\delta(K)}{-\log \delta}.$$

If the limits are equal, the common value \dim_B is referred to as the box-counting dimension of K . In particular, a smooth, compact m -dimensional submanifold of \mathbb{R}^d (or a subdomain of it) has \dim_B equal to m .

Remark 3.1. For a general bounded set K one has

$$\dim_H K \leq \underline{\dim}_B K \leq \overline{\dim}_B K, \quad (3.2)$$

where \dim_H denotes the Hausdorff dimension. Equalities hold for many regular sets, in particular for Ahlfors regular sets, i.e. sets which support a Borel regular measure μ such that, for some constant $C > 1$, and all $x \in K$,

$$C^{-1}r^s \leq \mu(B(x, r)) \leq Cr^s,$$

where the exponent s is the common dimension.

Lemma 3.2. Assume $K \subset \mathbb{R}^d$ is a compact set of positive lower box-counting dimension $m = \underline{\dim}_B K > 0$. For every $n \geq 2$, there exists a constant $A_n > 0$ such that

$$\forall f_n \in \mathcal{P}_n, \quad \|f_n\|_K \geq A_n,$$

and for n large enough, one may take $A_n = \exp(-2^\alpha n^{1+2\alpha/m})$.

Proof. Denote by $M_n(K)$ the smallest radius δ such that K can be covered by n closed balls of radius δ . Since $n \geq N_{M_n(K)}(K)$, one has, for n large enough,

$$\frac{\log n}{-\log M_n(K)} \geq \frac{\log N_{M_n(K)}(K)}{-\log M_n(K)} \geq \frac{m}{2},$$

or equivalently

$$M_n(K) \geq \left(\frac{1}{n}\right)^{2/m}.$$

Since n balls of radius $M_n(K)/2$ cannot cover K , we deduce that for $f_n(y) := e^{(-\sum_{i=2}^n \frac{1}{|y-x_i|^\alpha})}$,

$$\exists y \in K, \quad \forall i = 2, \dots, n, \quad |y - x_i| > \frac{M_n(K)}{2},$$

which implies

$$\forall f_n \in \mathcal{P}_n, \quad \|f_n\|_K \geq \exp\left(-\frac{2^\alpha n}{M_n(K)^\alpha}\right) \geq \exp(-2^\alpha n^{\frac{2\alpha}{m}+1}).$$

Since, by assumption, for every n , $M_n(K)$ is positive, the first inequality shows the existence of the constant A_n , which may be chosen as the first exponential. For n large enough, it may also be chosen as the second exponential, expressed in terms of the lower box-counting dimension m of K . \square

We next obtain a uniform Bernstein-type estimate for $f_n \in \mathcal{P}_n$.

Proposition 3.3. *Assume $K \subset \mathbb{R}^d$ is a compact set of positive lower box-counting dimension $m > 0$. Then,*

$$\forall n \geq 2, \quad \exists C_n > 0, \quad \forall f_n \in \mathcal{P}_n, \quad \|\nabla f_n\|_{2,K} := \left\| \left[\sum_{j=1}^n |\partial f_n / \partial y_j|^2 \right]^{1/2} \right\|_K \leq C_n \|f_n\|_K,$$

where, for n large enough, $C_n = C_\alpha n^\beta$ with a constant C_α depending on α only, and $\beta = 2 + 1/\alpha + 2\alpha/m + 2/m$. In particular, $C_n^{1/n} \rightarrow 1$ as $n \rightarrow \infty$.

Note that by (3.2) the above proposition applies as soon as K has positive Hausdorff dimension.

Proof. We estimate $\partial f_n / \partial y_1$ on \mathbb{R}^d . We have

$$\frac{\partial f_n}{\partial y_1}(y) = \alpha f_n(y) \sum_{j=2}^n \frac{y_1 - (x_j)_1}{|y - x_j|^{\alpha+2}},$$

so that

$$\left| \frac{\partial f_n}{\partial y_1}(y) \right| \leq \alpha f_n(y) \sum_{j=2}^n \frac{1}{|y - x_j|^{\alpha+1}}.$$

Denote by M_α the maximum of the function

$$\rho > 0 \rightarrow \frac{1}{\rho^{\alpha+1}} \exp\left(-\frac{1}{\rho^\alpha}\right).$$

Using the fact that $\exp\left(-\frac{1}{|y-x_j|^\alpha}\right) \leq 1$ for $j = 2, \dots, n$ so that

$$\exp\left(-\sum_{j=2}^n \frac{1}{|y-x_j|^\alpha}\right) \leq \exp\left(-\frac{1}{|y-x_j|^\alpha}\right),$$

we obtain the estimate

$$\left|\frac{\partial f_n}{\partial y_1}(y)\right| \leq \alpha \sum_{j=2}^n \exp\left(-\frac{1}{|y-x_j|^\alpha}\right) \frac{1}{|y-x_j|^{\alpha+1}} \leq (n-1)\alpha M_\alpha.$$

Thus

$$\forall y \in \mathbb{R}^d, \quad \|\nabla f_n(y)\|_2 = \left[\sum_{j=1}^n \left|\frac{\partial f_n}{\partial y_j}(y)\right|^2\right]^{1/2} \leq n\sqrt{d}\alpha M_\alpha. \quad (3.3)$$

For any $\lambda > 0$, we consider the functions

$$f_n(y) = \exp\left(-\sum_{j=2}^n \frac{1}{|y-x_j|^\alpha}\right), \quad F_n(y) = \exp\left(-\sum_{j=2}^n \frac{1}{|y-\lambda x_j|^\alpha}\right),$$

on K and λK . It is easily checked that, for $y \in K$,

$$F_n(\lambda y) = (f_n(y))^{\lambda^{-\alpha}}, \quad \lambda \nabla F_n(\lambda y) = \lambda^{-\alpha} (f_n(y))^{\lambda^{-\alpha}-1} \nabla f_n(y).$$

Thus,

$$\|\nabla f_n\|_{2,K} \leq \lambda^{\alpha+1} \|\nabla F_n\|_{2,\lambda K} \|f_n\|_K^{1-\lambda^{-\alpha}}.$$

We choose λ so as to minimize $\lambda^{\alpha+1} \|f_n\|_K^{-\lambda^{-\alpha}}$. One may check that the function

$$\lambda > 0 \rightarrow \lambda^{\alpha+1} \|f_n\|_K^{-\lambda^{-\alpha}}$$

has a unique minimum which is

$$\left(\frac{e\alpha \log(1/\|f_n\|_K)}{\alpha+1}\right)^{1+1/\alpha}.$$

Hence,

$$\|\nabla f_n\|_{2,K} \leq C_\alpha n \log(1/\|f_n\|_K)^{1+1/\alpha} \|f_n\|_K,$$

where we have used (3.3) applied to the function F_n . Making use of Lemma 3.2 gives the result. \square

For the next result, we need to apply the Bernstein estimate locally. Thus, we introduce a local box-counting dimension of a subset A of \mathbb{R}^d ,

$$\forall x \in A, \quad \dim_B(A, x) := \lim_{r \rightarrow 0} \dim_B(A \cap B(x, r)).$$

In the next theorem, we assume that the compact set K satisfies the following hypotheses:

1. There exists a $\rho > 0$ such that, for all $x \in K$, $\dim_B(K, x) \geq \rho$.
2. For $\delta = \delta(K)$ sufficiently small one can find $L = L(K, \delta) > 0$ so that for any $x \in K$ and $y \in K \cap B(x, \delta)$, there is a rectifiable curve $\gamma \subset K$ joining x to y of length at most $L|x - y|$ where L is independent of x, y . This property of a set is often called local quasiconvexity in the literature, see e.g. [8]. It implies in particular that K is locally path connected.

We can now prove:

Theorem 3.4. *Let μ be a positive measure on K of finite total mass and suppose μ satisfies the following mass density condition: there exist constants $T, c, r_0 > 0$ such that for all $x \in K$,*

$$\frac{\mu(B(x, r))}{r^T} \geq c > 0 \text{ for all } r \leq r_0. \quad (3.4)$$

Then for any $Q \in C(K)$,

$$\|f_n^Q\|_K \leq M_n \int_K f_n^Q(x) d\mu(x) \text{ for all } f_n^Q \in \mathcal{P}_n^Q \quad (3.5)$$

where $M_n = M_n(Q)$ satisfies $M_n^{1/n} \rightarrow 1$.

Proof. Fix $f_n^Q = f_n e^{-2nQ} \in \mathcal{P}_n^Q$ and let $w \in K$ be a point with

$$\|f_n e^{-2nQ}\|_K = f_n(w) e^{-2nQ(w)}.$$

Given $\epsilon > 0$ sufficiently small, there is a $\delta > 0$ such that

$$|Q(a) - Q(b)| \leq \epsilon \text{ if } a, b \in K \text{ with } |a - b| < \delta.$$

We have

$$f_n(w) e^{-2nQ(w)} \geq f_n(x) e^{-2nQ(x)} \text{ for all } x \in K.$$

Thus if $|x - w| < \delta$ we have

$$f_n(w) e^{2n\epsilon} \geq f_n(x)$$

and hence

$$\|f_n\|_{B(w, \delta)} \leq f_n(w) e^{2n\epsilon}. \quad (3.6)$$

For $x \in B(w, \delta)$,

$$f_n(x) - f_n(w) = \int_0^1 \nabla f_n(r(t)) r'(t) dt$$

where $t \rightarrow r(t)$ is a smooth curve joining w to x as above. Applying Lemma 3.3 in $B(w, \delta)$,

$$|f_n(x) - f_n(w)| \leq \|\nabla f_n\|_{2, B(w, \delta)} \int_0^1 \|r'(t)\|_2 dt$$

$$\leq \|\nabla f_n\|_{2, B(w, \delta)} L|x - w| \leq C_n L|x - w| \cdot \|f_n\|_{B(w, \delta)}.$$

Now for n large so that $e^{-3n\epsilon} < \min(\delta, r_0)$ and $C_n L \cdot e^{-n\epsilon} < 1/2$, for $x \in B(w, e^{-3n\epsilon}) \subset B(w, \delta)$ we may apply this estimate together with (3.6) to conclude that

$$|f_n(x) - f_n(w)| \leq C_n L \cdot e^{-3n\epsilon} \cdot \|f_n\|_{B(w, \delta)} \leq C_n L \cdot e^{-3n\epsilon} \cdot f_n(w) e^{2n\epsilon} \leq \frac{1}{2} f_n(w).$$

Hence

$$f_n(x) \geq \frac{1}{2} f_n(w) \text{ for large } n \text{ if } x \in B(w, e^{-3n\epsilon}).$$

Finally,

$$\begin{aligned} \int_K f_n(x) e^{-2nQ(x)} d\mu(x) &\geq \int_{B(w, e^{-3n\epsilon})} f_n(x) e^{-2nQ(x)} d\mu(x) \\ &\geq \frac{1}{2} f_n(w) e^{-2nQ(w)} e^{-2n\epsilon} \mu(B(w, e^{-3n\epsilon})) \\ &\geq \frac{1}{2} f_n(w) e^{-2nQ(w)} e^{-2n\epsilon} c e^{-3nT\epsilon} \geq \|f_n e^{-2nQ}\|_K \cdot \frac{c}{2} e^{-n\epsilon(2+3T)}. \end{aligned}$$

□

Remark 3.5. We call a measure μ satisfying (3.5) for each $Q \in C(K)$ a *strong Bernstein-Markov measure* (for Riesz potentials) on K . As an example, for K a smooth, compact m -dimensional submanifold of \mathbb{R}^d , the Hausdorff m -measure (or equivalently its volume form) is a strong Bernstein-Markov measure on K . A special case of the results in this section was proved in [1].

4 Free energy asymptotics and a.s. convergence

Let $K \subset \mathbb{R}^d$ be compact and of positive Riesz capacity; $Q \in C(K)$; and fix a measure μ on K satisfying (3.5). For each $n = 2, 3, \dots$, define

$$Z_n = Z_n(K, Q, \mu) := \int_{K^n} VDM_n^Q(x_1, x_2, \dots, x_n) d\mu(x_1) \cdots d\mu(x_n).$$

Proposition 4.1. *With K, Q and μ as above,*

$$\lim_{n \rightarrow \infty} Z_n^{1/n^2} = \delta^Q(K) = \exp(-V_w). \quad (4.1)$$

Proof. We make use of inequality (3.5). Fix a set of n points $a_1, \dots, a_n \in K$ with

$$\max_{x_1, \dots, x_n \in K} VDM_n^Q(x_1, x_2, \dots, x_n) = VDM_n^Q(a_1, a_2, \dots, a_n).$$

The function

$$y \rightarrow VDM_n^Q(y, a_2, \dots, a_n)$$

is, up to a multiplicative constant, a function in \mathcal{P}_n^Q which attains its maximum value on K at $y = a_1$. Using (3.5),

$$VDM_n^Q(a_1, a_2, \dots, a_n) \leq M_n \int_K VDM_n^Q(y, a_2, \dots, a_n) d\mu(y).$$

Now consider, for each fixed $y \in K$,

$$z \rightarrow VDM_n^Q(y, z, a_3, \dots, a_n).$$

This is, up to a multiplicative constant, a function in \mathcal{P}_n^Q ; and

$$VDM_n^Q(y, a_2, a_3, \dots, a_n) \leq \max_{z \in K} VDM_n^Q(y, z, a_3, \dots, a_n).$$

Using (3.5) again,

$$VDM_n^Q(y, a_2, a_3, \dots, a_n) \leq M_n \int_K VDM_n^Q(y, z, a_3, \dots, a_n) d\mu(z)$$

which gives

$$VDM_n^Q(a_1, a_2, \dots, a_n) \leq M_n^2 \int_K \int_K VDM_n^Q(y, z, a_3, \dots, a_n) d\mu(z) d\mu(y).$$

Repeating this argument $n - 2$ times gives

$$VDM_n^Q(a_1, a_2, \dots, a_n) \leq M_n^n Z_n.$$

On the other hand,

$$Z_n \leq VDM_n^Q(a_1, a_2, \dots, a_n) \mu(K)^n.$$

Combining these last two displayed inequalities with 2. of Theorem 2.9 and the fact that $M_n^{1/n} \rightarrow 1$ gives the result. □

We define a probability measure $Prob_n$ on K^n as follows: for a Borel set $A \subset K^n$,

$$Prob_n(A) := \frac{1}{Z_n} \cdot \int_A |VDM_n^Q(x_1, \dots, x_n)|^2 d\mu(x_1) \cdots d\mu(x_n). \quad (4.2)$$

This coincides with (1.1). From Proposition 4.1 we obtain the following estimate.

Corollary 4.2. *With K, Q and μ as above, given $\eta > 0$, define*

$$A_{n,\eta} := \{(x_1, \dots, x_n) \in K^n : VDM_n^Q(x_1, \dots, x_n) \geq (\delta^Q(K) - \eta)^{n^2}\}. \quad (4.3)$$

There exists $n^ = n^*(\eta)$ such that for all $n > n^*$,*

$$Prob_n(K^n \setminus A_{n,\eta}) \leq \left(1 - \frac{\eta}{2\delta^Q(K)}\right)^{n^2} \mu(K)^n.$$

We get the induced product probability measure \mathbf{P} on the space of arrays on K ,

$$\chi := \{X = \{\mathbf{Z}_n := (x_{n1}, \dots, x_{nn}) \in K^n\}_{n \geq 1}\},$$

namely,

$$(\chi, \mathbf{P}) := \prod_{n=1}^{\infty} (K^n, Prob_n).$$

From standard arguments using the Borel-Cantelli lemma, we obtain:

Corollary 4.3. *With K, Q and μ as above, for \mathbf{P} -a.e. array $X = \{\mathbf{Z}_n\} \in \chi$,*

$$\mu_n := \sum_{j=1}^n \delta_{x_{nj}} \rightarrow \mu_{K,Q} \text{ weak-}^* \text{ as } n \rightarrow \infty.$$

Define the probability measures (*one-point correlation functions*)

$$d\tau_n(x) := \frac{1}{Z_n} R_1^{(n)}(x) d\mu(x)$$

where

$$R_1^{(n)}(x) := \int_{K^{n-1}} |VDM_n^Q(x, x_2, \dots, x_n)|^2 d\mu(x_2) \cdots d\mu(x_n).$$

Using Corollary 4.3, we get the following deterministic result.

Corollary 4.4. *With K, Q and μ as above,*

$$\tau_n \rightarrow \mu_{K,Q} \text{ as } n \rightarrow \infty.$$

Proof. For $f \in C(K)$, we show $\int_K f d\tau_n \rightarrow \int_K f d\mu_{K,Q}$. Writing $\mathbf{Z}_n = \{x_{nj}\}$ and $\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{x_{nj}}$, given $\epsilon > 0$, let

$$E_n := \{X = \{\mathbf{Z}_n\} \in \chi : \left| \int_K f d\mu_n - \int_K f d\mu_{K,Q} \right| \geq \epsilon\}$$

and $F_n := \{\mathbf{Z}_n \in K^n : \left| \int_K f d\mu_n - \int_K f d\mu_{K,Q} \right| \geq \epsilon\}$. From Corollary 4.3,

$$Prob_n(F_n) = \mathbf{P}(E_n) \leq \mathbf{P}(\cup_{m \geq n} E_m) \downarrow 0.$$

Hence $Prob_n(F_n) < \epsilon$ for n large. Splitting K^n into F_n and $K^n \setminus F_n$ we obtain

$$|\int_{K^n} (\int_K f d\mu_n - \int_K f d\mu_{K,Q}) dProb_n(\mathbf{Z}_n)| < \epsilon + 2\|f\|_K \cdot \epsilon.$$

Thus

$$\int_{K^{n+1}} \int_K f d\mu_n dProb_n(\mathbf{Z}_n) \rightarrow \int_K f d\mu_{K,Q}.$$

Using $\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{x_{nj}}$,

$$\begin{aligned} & \int_{K^n} \int_K f d\mu_n dProb_n(\mathbf{Z}_n) = \\ & \frac{1}{Z_n} \int_{K^n} \left(\frac{1}{n} \sum_{j=1}^n f(x_{nj}) \right) |VDM_n^Q(x_{n1}, \dots, x_{nn})| \prod_{j=1}^n d\mu(x_{nj}) \\ & = \frac{1}{Z_n} \int_K f(x_{n1}) \left(\int_{K^{n-1}} |VDM_n^Q(x_{n1}, \dots, x_{nn})| \prod_{j=2}^n d\mu(x_{nj}) \right) d\mu(x_{n1}) \\ & = \frac{1}{Z_n} \int_K f(x_{n1}) R_1^{(n)}(x_{n1}) d\mu(x_{n1}) = \int_K f d\tau_n. \end{aligned}$$

□

5 Large deviation principle

We will need an approximation lemma to prove our large deviation result.

Lemma 5.1. *Let $K \subset \mathbb{R}^d$ be compact and of positive Riesz capacity and let $\tau \in \mathcal{M}(K)$ with $I(\tau) < \infty$. There exist an increasing sequence of compact sets K_m in K , a sequence of functions $\{Q_m\} \subset C(K)$, and a sequence of measures $\tau_m \in \mathcal{M}(K_m)$ satisfying*

1. *the measures τ_m tend weakly to τ , as $m \rightarrow \infty$;*
2. *the energies $I(\tau_m)$ tend to $I(\tau)$ as $m \rightarrow \infty$;*
3. *the measures τ_m are equal to the weighted equilibrium measures τ_{K,Q_m} .*

Proof. By Lusin's continuity theorem applied in K , for every integer $m \geq 1$, there exists a compact subset K_m of K such that $\tau(K \setminus K_m) \leq 1/m$ and $U^\tau|_{K_m}$ is continuous on K_m . We may assume that K_m is increasing as m tends to infinity. Then the measures $\tilde{\tau}_m := \tau|_{K_m}$ are increasing and tend weakly to τ . We have

$$\chi_m(x, y) \frac{1}{|x - y|^\alpha} \uparrow \frac{1}{|x - y|^\alpha} \text{ as } m \rightarrow \infty$$

$(\tau \times \tau)$ -almost everywhere on $K \times K$ where $\chi_m(x, y)$ is the characteristic function of $K_m \times K_m$ and we agree that the left-hand sides vanish when $x = y \notin K_m$. Hence, by monotone convergence we have

$$I(\tilde{\tau}_m) \rightarrow I(\tau) \quad \text{as } m \rightarrow \infty.$$

Next, define $\tau_m := \tilde{\tau}_m / \tau(K_m)$. Clearly we have $I(\tau_m) \rightarrow I(\tau)$ since $I(\tilde{\tau}_m) \rightarrow I(\tau)$ and $\tau(K_m) \uparrow 1$. Define, for $x \in K$,

$$Q_m(x) := -U^{\tau_m}(x) = \frac{-1}{\tau(K_m)} \cdot U^{\tilde{\tau}_m}(x).$$

We first show Q_m is continuous on K_m . Since U^{τ_m} is lower semicontinuous, it suffices to show it is upper semicontinuous. This follows since $U^{\tau - \tilde{\tau}_m} = U^\tau - U^{\tilde{\tau}_m}$ is lower semicontinuous (indeed, continuous) and U^τ is continuous on K_m . By the continuity property of Riesz potentials ([9], Theorem 1.7, p. 69, valid for all $0 < \alpha < d$), we have $Q_m = -U^{\tau_m}$ is continuous on \mathbb{R}^d (and in particular on K). Item 3. follows from Remark 2.5. \square

We have all of the ingredients needed to follow the arguments of section 6 of [3] to prove the analogue of Theorem 6.6 there and hence a large deviation principle (Definition 5.5 and Theorem 5.7 below) which quantifies the statement of \mathbf{P} -a.e. convergence for arrays $X = \{\mathbf{Z}_n\}$ where $\mathbf{Z}_n = \{x_{nj}\}$ of $\frac{1}{n} \sum_{j=1}^n \delta_{x_{nj}}$ to $\mu_{K,Q}$. Given $G \subset \mathcal{M}(K)$, for each $n = 1, 2, \dots$ we let

$$\tilde{G}_n := \{\mathbf{a} = (a_1, \dots, a_n) \in K^n, \frac{1}{n} \sum_{j=1}^n \delta_{a_j} \in G\}, \quad (5.1)$$

and set

$$J_n^Q(G) := \left[\int_{\tilde{G}_n} |VDM_n^Q(\mathbf{a})| d\nu(\mathbf{a}) \right]^{1/n^2}. \quad (5.2)$$

Definition 5.2. For $\mu \in \mathcal{M}(K)$ we define

$$\overline{J}^Q(\mu) := \inf_{G \ni \mu} \overline{J}^Q(G) \quad \text{where} \quad \overline{J}^Q(G) := \limsup_{n \rightarrow \infty} J_n^Q(G);$$

$$\underline{J}^Q(\mu) := \inf_{G \ni \mu} \underline{J}^Q(G) \quad \text{where} \quad \underline{J}^Q(G) := \liminf_{n \rightarrow \infty} J_n^Q(G)$$

where the infimum is taken over all open neighborhoods $G \subset \mathcal{M}(K)$ of μ . If $Q = 0$ we simply write $\overline{J}(\mu), \underline{J}(\mu)$.

We will only consider weights $Q \in C(K)$; thus the analogue of Lemma 6.3 in [3] is simpler:

Lemma 5.3. *The following properties hold (and with the $\underline{J}, \underline{J}^Q$ functionals as well):*

1. $\overline{J}^Q(\mu) \leq e^{-I^Q(\mu)}$ for $Q \in C(K)$;

2. $\overline{J}^Q(\mu) = \overline{J}(\mu) \cdot e^{-2 \int_K Q d\mu}$ for $Q \in C(K)$.

Following the steps in section 6 of [3] with Corollary 5.3 there replaced by our approximation result, Lemma 5.1, we obtain equality of the \overline{J}^Q and \underline{J}^Q functionals for any admissible weight Q provided ν is a strong Bernstein-Markov measure for K (see Theorem 6.6 in [3]). Note because of 2. in Lemma 5.3 we do not need the general Lemma 5.2 of [3].

Theorem 5.4. *Let $K \subset \mathbb{R}^d$ be a compact set of positive Riesz capacity. Let $\nu \in \mathcal{M}(K)$ be a strong Bernstein-Markov measure for K (e.g., if K and ν satisfy the hypotheses of Theorem 3.4).*

(i) *For any $\mu \in \mathcal{M}(K)$,*

$$\log \overline{J}(\mu) = \log \underline{J}(\mu) = -I(\mu).$$

(ii) *Let $Q \in C(K)$. Then for any $\mu \in \mathcal{M}(K)$,*

$$\overline{J}^Q(\mu) = \overline{J}(\mu) \cdot e^{-2 \int_K Q d\mu}$$

(and with the $\underline{J}, \underline{J}^Q$ functionals as well) so that,

$$\log \overline{J}^Q(\mu) = \log \underline{J}^Q(\mu) = -I^Q(\mu). \quad (5.3)$$

Thus we simply write J, J^Q without an underline or overline.

Define $j_n : K^n \rightarrow \mathcal{M}(K)$ via

$$j_n(x_1, \dots, x_n) = \frac{1}{n} \sum_{j=1}^n \delta_{x_j}. \quad (5.4)$$

The push-forward $\sigma_n := (j_n)_*(\text{Prob}_n)$ is a probability measure on $\mathcal{M}(K)$: for a Borel set $G \subset \mathcal{M}(K)$,

$$\sigma_n(G) = \frac{1}{Z_n} \int_{\tilde{G}_n} |VDM_n^Q(x_1, \dots, x_n)| d\nu(x_1) \cdots d\nu(x_n). \quad (5.5)$$

Definition 5.5. The sequence $\{\sigma_n\}$ of probability measures on $\mathcal{M}(K)$ satisfies a **large deviation principle** (LDP) with good rate function \mathcal{I} and speed $\{s_n\}$ with $s_n \rightarrow \infty$ if for all measurable sets $\Gamma \subset \mathcal{M}(K)$,

$$-\inf_{\mu \in \Gamma^0} \mathcal{I}(\mu) \leq \liminf_{n \rightarrow \infty} \frac{1}{s_n} \log \sigma_n(\Gamma) \text{ and} \quad (5.6)$$

$$\limsup_{n \rightarrow \infty} \frac{1}{s_n} \log \sigma_n(\Gamma) \leq -\inf_{\mu \in \overline{\Gamma}} \mathcal{I}(\mu). \quad (5.7)$$

On $\mathcal{M}(K)$, to prove a LDP it suffices to work with a base for the weak topology. The following is a special case of a basic general existence result, Theorem 4.1.11 in [6].

Proposition 5.6. *Let $\{\sigma_\epsilon\}$ be a family of probability measures on $\mathcal{M}(K)$. Let \mathcal{B} be a base for the topology of $\mathcal{M}(K)$. For $\mu \in \mathcal{M}(K)$ let*

$$\mathcal{I}(\mu) := - \inf_{\{G \in \mathcal{B}: \mu \in G\}} \left(\liminf_{\epsilon \rightarrow 0} \epsilon \log \sigma_\epsilon(G) \right).$$

Suppose for all $\mu \in \mathcal{M}(K)$,

$$\mathcal{I}(\mu) := - \inf_{\{G \in \mathcal{B}: \mu \in G\}} \left(\limsup_{\epsilon \rightarrow 0} \epsilon \log \sigma_\epsilon(G) \right).$$

Then $\{\sigma_\epsilon\}$ satisfies a LDP with rate function $\mathcal{I}(\mu)$ and speed $1/\epsilon$.

Following section 7 of [3], Theorem 5.4 and Proposition 5.6 immediately yield a large deviation principle:

Theorem 5.7. *Assume ν is a strong Bernstein-Markov measure for K and $Q \in C(K)$. The sequence $\{\sigma_n = (j_n)_*(\text{Prob}_n)\}$ of probability measures on $\mathcal{M}(K)$ satisfies a large deviation principle with speed n^2 and good rate function $\mathcal{I} := \mathcal{I}_{K,Q}$ where, for $\mu \in \mathcal{M}(K)$,*

$$\mathcal{I}(\mu) := \log J^Q(\mu_{K,Q}) - \log J^Q(\mu) = I^Q(\mu) - I^Q(\mu_{K,Q}).$$

6 Open problems

We conclude with some questions which are motivated by logarithmic weighted potential theory as in [11].

1. Where does the supremum norm of a function $f_n^Q \in \mathcal{P}_n^Q$ (see (3.1)) live? More precisely, can one control the growth of f_n^Q from its size on $S_w = \text{supp}(\mu_{K,Q})$ and/or on $S_w^* := \{x \in K : U^{\mu_{K,Q}}(x) + Q(x) \leq F_w\}$?
2. Let K be a compact set of positive Riesz capacity. Does there exist a positive measure μ on K of finite total mass satisfying (3.5) for $Q = 0$? Can such a μ be discrete? See [1] for related results.
3. Generalize the results in this paper to the case where $K \subset \mathbb{R}^d$ is allowed to be a closed, unbounded set for appropriate weights Q ([5] considers $K = \mathbb{R}^d$).

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